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A NUMERICAL COMPARISON OF FINITE DIFFERENCE AND LINEAR SHOOTING METHODS FOR A NON-HOMOGENEOUS CAUCHY-EULER BOUNDARY VALUE PROBLEM

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Abstract: *This study presents a comparative numerical analysis of the Finite Difference Method (FDM) and the Linear Shooting Method (LSM) for solving a second-order non-homogeneous Cauchy-Euler boundary value problem subject to Dirichlet, Neumann, and Robin boundary conditions. In contrast to previous studies that often employ different differential equations for different numerical experiments, this study uses an identical Cauchy-Euler equation while varying only the boundary conditions. This unified framework enables a more systematic investigation of the influence of boundary conditions on the performance of the numerical methods. Numerical solutions are computed using three mesh sizes,* $N = 10$, $N = 20$, and $N = 50$. *The accuracy of the methods is assessed by comparing the numerical solutions with the exact solution using Maximum Absolute Error (MaxAE) and Mean Absolute Error (MAE). In addition, graphical comparisons of the exact and numerical solutions, together with their corresponding error distributions, are presented to illustrate the solution behavior under different boundary conditions. The numerical results show that both methods produce accurate approximations with decreasing errors as the mesh is refined. The FDM exhibits approximately second-order convergence and requires less execution time, whereas the LSM achieves a higher convergence order and consistently produces smaller MaxAE and MAE values, indicating superior numerical accuracy.*

Keywords: *Cauchy-Euler equation; Finite Difference Method; Linear Shooting Method*

Abstrak: Penelitian ini menyajikan analisis numerik komparatif antara Metode Beda Hingga (FDM) dan Metode *Shooting* Linear (LSM) untuk menyelesaikan masalah nilai batas (BVP) *Cauchy-Euler* non-homogen orde dua dengan kondisi batas *Dirichlet, Neumann,* dan *Robin*. Berbeda dengan penelitian sebelumnya yang umumnya menggunakan persamaan diferensial yang berbeda pada setiap eksperimen numerik, penelitian ini menggunakan persamaan *Cauchy-Euler* yang sama dengan memvariasikan kondisi batasnya. Pendekatan ini memungkinkan pengkajian yang lebih sistematis mengenai pengaruh kondisi batas terhadap performa metode numerik yang digunakan. Solusi numerik dihitung menggunakan tiga ukuran mesh, yaitu $N = 10$, $N = 20$, dan $N=50$. Tingkat akurasi kedua metode dievaluasi dengan membandingkan solusi numerik terhadap solusi eksak menggunakan *Maximum Absolute Error* (MaxAE) dan *Mean Absolute Error* (MAE). Selain itu, perbandingan grafis antara solusi eksak dan solusi numerik beserta distribusi galatnya

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juga disajikan untuk menggambarkan perilaku solusi pada masing-masing kondisi batas. Hasil numerik menunjukkan bahwa kedua metode mampu memberikan pendekatan yang akurat dengan galat yang semakin kecil seiring dengan penyempurnaan ukuran mesh. FDM memiliki orde konvergensi sekitar dua dan memerlukan waktu komputasi yang lebih singkat, sedangkan LSM memiliki orde konvergensi yang lebih tinggi serta secara konsisten menghasilkan nilai MaxAE dan MAE yang lebih kecil, yang menunjukkan tingkat akurasi numerik yang lebih baik.

Kata kunci: Persamaan *Cauchy-Euler*; Metode Beda Hingga; Metode *Shooting* Linear

INTRODUCTION

Mathematical modeling plays a fundamental role in describing and predicting phenomena arising in science, engineering, and applied mathematics. Many physical systems, such as heat conduction, structural deformation, fluid flow, and diffusion processes, are governed by differential equations whose solutions characterize the behavior of the underlying systems (Yang, 2024). Consequently, the accurate solution of differential equations remains an important topic in both theoretical and computational mathematics.

Boundary value problems (BVPs) may be classified as linear or nonlinear depending on the form of the differential equation (Krishnamohan et al., 2025). Among the various classes of BVPs, second-order linear boundary value problems are of particular importance because many physical laws equilibrium and steady-state phenomena can be expressed in terms of second-order differential equations. Such problems are commonly associated with Dirichlet, Neumann, and Robin boundary conditions, which represent different physical constraints imposed on the solution (Adak & Mandal, 2022). An important class of second-order linear BVPs is the Cauchy-Euler equation, which is characterized by variable coefficients that are powers of the independent variable. This type of equation frequently arises in scientific and engineering applications and presents additional computational challenges compared with constant-coefficient equations (Tavangar, 2024). Although analytical solutions can be obtained for certain classes of problems, exact solutions are often unavailable or difficult to derive in practical applications. Consequently, numerical approximation methods have become indispensable tools for obtaining accurate and efficient solutions (Wadho et al., 2025). The development, analysis, and comparison of reliable numerical methods therefore remain important topics in computational mathematics and scientific computing.

Several numerical methods have been proposed for solving second-order linear BVPs. Comparative studies conducted by Markus et al. (2018), Masenge & Malaki (2020), and Alhassan et al. (2023) examined the performance of the FDM and the Shooting Method based on the classical fourth-order Runge-Kutta (RK4) scheme for solving linear Dirichlet BVPs. The application of FDM to Dirichlet boundary value problems was also investigated by Mukhtarov et al. (2019), who obtained numerical solutions using a finite difference discretization approach. Meanwhile, Edun & Akinlabi (2021) investigated the application of the Shooting Method using Euler integration. The applicability of FDM was further explored by Adak & Mandal (2022),

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https://ejurnal.unisap.ac.id/leibniz/index

who developed a finite difference framework for solving second-order boundary value problems under Dirichlet, Neumann, and Robin boundary conditions. Dung & Quang (2024) also examined finite difference approaches with high-order accuracy for Dirichlet and Robin BVPs. For Robin boundary conditions, Habibah et al. (2020) studied the numerical solution of Cauchy-Euler boundary value problems using both the Finite Difference Method and the Shooting Method. In addition, Habibah & Mulyanti (2023) analyzed the accuracy of the Shooting Method for solving Sturm-Liouville boundary value problems. More recently, Kamelia et al. (2025) extended the work of Masenge & Malaki (2020) by incorporating Neumann boundary conditions and the Collocation Method, thereby broadening the scope of the comparative analysis. Furthermore, Ardiana et al. (2025) investigated the performance of the FDM and the Shooting Method in solving second-order linear boundary value problems under Dirichlet, Neumann, and Robin boundary conditions, providing a broader assessment of these numerical approaches across different types of boundary value problems.

Despite these developments, most comparative studies have focused on specific types of boundary conditions or have employed different differential equations for numerical experiments. Consequently, the influence of boundary conditions on the performance of numerical methods has not been thoroughly investigated under a unified problem setting. In particular, comparative analyses employing the same non-homogeneous Cauchy-Euler differential equation with different boundary conditions remain limited. Such a unified framework is important because it isolates the effect of boundary conditions on numerical accuracy and solution behavior, allowing a more objective evaluation of the numerical methods. Unlike previous studies that mainly compare numerical methods using different test equations, this study employs an identical differential equation for all numerical experiments so that the effect of boundary conditions on the accuracy of the Finite Difference Method (FDM) and Linear Shooting Method (LSM) can be investigated more systematically. Because the Cauchy-Euler differential equation considered in this study is linear, the shooting approach is implemented in the form of the Linear Shooting Method (LSM), which is specifically designed for linear boundary value problems.

Therefore, this study aims to compare the performance of FDM and the LSM in solving a second-order non-homogeneous Cauchy-Euler boundary value problem subject to Dirichlet, Neumann, and Robin boundary conditions. The comparison is conducted using the same differential equation while varying only the boundary conditions. Numerical performance is evaluated in terms of *Maximum Absolute Error (MaxAE)*, Mean Absolute Error (MAE), estimated order of convergence (EOC), and computational cost (execution time) for different mesh sizes. In addition, graphical comparisons between the exact and numerical solutions are presented to illustrate the accuracy and behavior of both methods under different boundary conditions.

Leibniz: Jurnal Matematika
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Vol. 6 No. 02 Hal. 01-15
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RESEARCH METHOD

This study compares the performance of the Finite Difference Method (FDM) and the Linear Shooting Method (LSM) in solving a non-homogeneous Cauchy-Euler boundary value problem with Dirichlet, Neumann, and Robin boundary conditions. Numerical solutions are obtained for several mesh sizes and are evaluated using Maximum Absolute Error (MaxAE), Mean Absolute Error (MAE), estimated order of convergence, and computational cost. The numerical computations were implemented using MATLAB R2019a. The overall research procedure is illustrated in Figure 1.

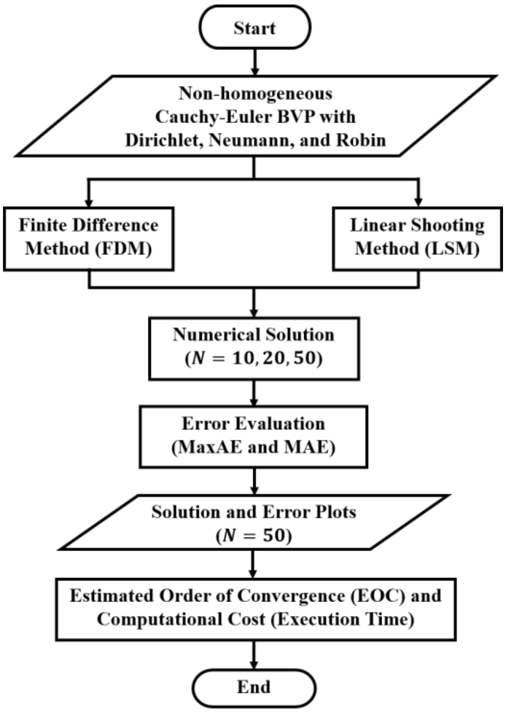

Figure 1. Flowchart of The Numerical Procedure

Boundary Value Problem

Consider the second-order Cauchy-Euler equation

$$pt^2x''(t) + qtx'(t) + rx(t) = f(t), \quad a \le t \le b, \tag{1}$$

where p, q, r are constant and $f(t)$ is a given function of the independent variable t. The equation is said to be homogeneous when $f(t) = 0$ and non-homogeneous when $f(t) \neq 0$ (Tavangar, 2024). The BVP is obtained by supplementing the differential equation with one of the following types of boundary conditions:

Dirichlet : $x(a) = \alpha,\ x(b) = \beta,$

Neumann : $x'(a) = \alpha,\ x'(b) = \beta,$

Robin : $\alpha_1 x(a) + \alpha_2 x'(a) = \gamma_1, \quad \beta_1 x(b) + \beta_2 x'(b) = \gamma_2,$

where $\alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, and\ \gamma_2$ are prescribed constants.


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The problem considered in this study is

$$x''(t)+\frac{2}{t}\,x'(t)-\frac{2}{t^2}\,x(t)=\frac{sin(\ln t)}{t^2},\qquad 1\le t\le 2 \tag{2}$$

subject to the following three types of boundary conditions:

Dirichlet : $x(1)=1,\qquad x(2)=2,$

Neumann : $x'(1)=0,\qquad x'(2)=1,$

Robin : $x(1)+x'(1)=1,\qquad x(2)+x'(2)=2.$

General Solution

The exact solution of equation (2) is derived by combining the homogeneous and particular solutions. The corresponding homogeneous equation is

$$x''+\frac{2}{t}x'-\frac{2}{t^2}x=0,$$

Substituting the ansatz $x=t^a$, $x'=at^{a-1}$, $x''=a(a-1)t^{a-2}$ yields the indicial equation $(a^2+a-2)t^{a-2}=0$, giving roots $a_1=1$ and $a_2=-2$, thus the homogeneous solution is

$$x_h=c_1t+\frac{c_2}{t^2}.$$

Assume particular solution of the form

$$x_p=k_1\sin(\ln t)+k_2\cos(\ln t),\qquad x_p'=k_1\frac{\cos(\ln t)}{t}-k_2\frac{\sin(\ln t)}{t},$$

$$x_p''=(-k_1+k_2)\frac{\sin(\ln t)}{t^2}+(-k_1-k_2)\frac{\cos(\ln t)}{t^2}.$$

Then, substituting into (2) and equating coefficients gives $k_1=-\frac{3}{10}$ and $k_2=-\frac{1}{10}$, so that particular solution is

$$x_p=-\frac{3}{10}\sin(\ln t)-\frac{1}{10}\cos(\ln t).$$

The general solution is therefore $x(t)=x_h+x_p$,

$$x(t)\quad=c_1t+\frac{c_2}{t^2}-\frac{3}{10}\sin(\ln t)-\frac{1}{10}\cos(\ln t) \tag{3}$$

Applying each set of boundary conditions to (3) yields the corresponding constants c_1 and c_2, which determine the exact solution for each case.

Finite Difference Method

The interval $a\le t\le b$ in equation (2) is divided into N equal subintervals of length $\Delta t=h$, resulting in a set of uniformly spaced grid points

$$t_i=a+ih,\quad h=\frac{b-a}{N},\qquad i=0,1,2\ldots,N. \tag{4}$$

The numerical solution is approximated at each grid point t_i where x_i denotes the approximation of $x(t_i)$ for $i=0,\ldots,N$. The first and second derivatives are approximated using the central difference formulas of order $O(h^2)$

$$x'(t_i)=\frac{x_{i+1}-x_{i-1}}{2h} \quad \text{and} \quad x''(t_i)=\frac{x_{i+1}-2x_i+x_{i-1}}{h^2} \tag{5}$$

Substituting (5) into equation (2) at the interior grid points t_i gives

$$\frac{x_{i+1}-2x_i+x_{i-1}}{h^2}+\frac{2}{t_i}\left(\frac{x_{i+1}-x_{i-1}}{2h}\right)-\frac{2}{t_i^2}x_i=\frac{\sin(\ln t_i)}{t_i^2}, \tag{6}$$

which upon multiplication by h^2 and rearranging the terms, becomes

$$\left(1-\frac{h}{t_i}\right)x_{i-1}+\left(-2-\frac{2h^2}{t_i^2}\right)x_i+\left(1+\frac{h}{t_i}\right)x_{i+1}=\frac{h^2\sin(\ln t_i)}{t_i^2}. \tag{7}$$

Let $a_i=1-\frac{h}{t_i}$, $\quad b_i=-2-\frac{2h^2}{t_i^2}$, $\quad c_i=1+\frac{h}{t_i}$, then equation (7) can be rewritten as

$$a_i x_{i-1}+b_i x_i+c_i x_{i+1}=\frac{h^2\sin(\ln t_i)}{t_i^2}, \quad i=1,2\dots,N-1. \tag{8}$$

1. Dirichlet Boundary Value Problem

For the Dirichlet boundary conditions, the boundary values $x_0 = x(1) = 1$ and $x_N = x(2) = 2$ are directly incorporated into equation (8) for $i = 1$ dan $i = N-1$, respectively. This leads to a tridiagonal linear system $\boldsymbol{Px} = Q$, where P is a tridiagonal matrix of dimension $(N-1)\times(N-1)$ and Q is a vector of dimension $(N-1)\times(N-1)$, given by

$$\begin{pmatrix} b_1 & c_1 & 0 & & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & a_{N-2} & b_{N-2} & c_{N-2} \\ 0 & 0 & 0 & & 0 & a_{N-1} & b_{N-1} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{pmatrix}=\begin{pmatrix} \frac{h^2\sin(\ln t_1)}{t_1^2} \\ \frac{h^2\sin(\ln t_2)}{t_2^2} \\ \frac{h^2\sin(\ln t_3)}{t_3^2} \\ \vdots \\ \frac{h^2\sin(\ln t_{N-2})}{t_{N-2}^2} \\ \frac{h^2\sin(\ln t_{N-1})}{t_{N-1}^2} \end{pmatrix}. \tag{9}$$

The numerical solution at the interior nodes $x_1, x_2, \dots, x_{N-1}$ is obtained as $\boldsymbol{x} = P^{-1}Q$.

2. Neumann Boundary Value Problem

For the Neumann boundary conditions $x'(1) = 1$ and $x'(2) = 2$, the derivatives at the boundary nodes $i = 0$ and $i = N$ are approximated using the central difference formula, introducing ghost points x_{-1} and x_{N+1}

$$x_0' = x'(1) = 0 \quad \Rightarrow \quad \frac{x_1-x_{-1}}{2h}=0 \qquad\qquad x_N' = x'(2) = 1 \quad \Rightarrow \quad \frac{x_{N+1}-x_{N-1}}{2h}=1$$

$$x_1 = x_{-1} \qquad\qquad x_{N+1} = x_{N-1} + 2h$$

Substituting these ghost points into equation (8) at $i = 0$ and $i = N$ yields a tridiagonal linear system $P\boldsymbol{x} = Q$, where P and Q have dimension $(N+1)\times(N+1)$ and $(N+1)\times 1$, respectively, given by

$$\begin{pmatrix} b_0 & a_0+c_0 & 0 & & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & b_2 & & 0 & 0 & 0 \\ \vdots & & \ddots & & & \vdots & \\ 0 & 0 & 0 & \cdots & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & & 0 & a_N+c_N & b_N \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} = \begin{pmatrix} \frac{h^2\sin(\ln t_0)}{t_0^2} \\ \frac{h^2\sin(\ln t_1)}{t_1^2} \\ \frac{h^2\sin(\ln t_2)}{t_2^2} \\ \vdots \\ \frac{h^2\sin(\ln t_{N-1})}{t_{N-1}^2} \\ \frac{h^2\sin(\ln t_N)}{t_N^2} - 2hc_N \end{pmatrix}. \tag{10}$$

The numerical solution of the boundary value problem can thus be obtained by solving the linear system $\boldsymbol{x} = P^{-1}Q$.

3. Robin Boundary Value Problem

For the Robin boundary conditions $x(1) = x_0$, $x(2) = x_N$, the derivative terms are approximated using the central difference formula, introducing ghost points x_{-1} and x_{N+1}.

$$x_0 + x_0' = 1 \qquad\qquad x_N' = x'(2) = 2$$

$$x_0 + \frac{x_1 - x_{-1}}{2h} = 1 \qquad\qquad x_N + \frac{x_{N+1} - x_{N-1}}{2h} = 2$$

$$2hx_0 + x_1 - x_{-1} = 2h \qquad\qquad 2hx_N + x_{N+1} - x_{N-1} = 4h$$

$$x_{-1} = 2hx_0 + x_1 - 2h \qquad\qquad x_{N+1} = 4h + x_{N-1} - 2hx_N$$

Substituting the ghost points into equation (8) at $i = 0$ and $i = N$ yields a tridiagonal linear system $P\boldsymbol{x} = Q$, where P and Q have dimension $(N+1) \times (N+1)$ and $(N+1) \times 1$, respectively, given by

$$\begin{pmatrix} 2ha_0+b_0 & a_0+c_0 & 0 & & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & b_2 & & 0 & 0 & 0 \\ \vdots & & \ddots & & & \vdots & \\ 0 & 0 & 0 & \cdots & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & & 0 & a_N+c_N & b_N - 2hc_N \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} = \begin{pmatrix} \frac{h^2\sin(\ln t_0)}{t_0^2} + 2ha_0 \\ \frac{h^2\sin(\ln t_1)}{t_1^2} \\ \frac{h^2\sin(\ln t_2)}{t_2^2} \\ \vdots \\ \frac{h^2\sin(\ln t_{N-1})}{t_{N-1}^2} \\ \frac{h^2\sin(\ln t_N)}{t_N^2} - 4hc_N \end{pmatrix}. \tag{11}$$

The numerical solution is obtained as $\boldsymbol{x} = P^{-1}Q$.

Linear Shooting Method

1. Dirichlet Boundary Value Problem

Equation (2) with the Dirichlet boundary conditions $x(1) = 1$ and $x(2) = 2$ is solved by considering two auxiliary initial value problems.

$$u'' + \frac{2}{t}u' - \frac{2}{t^2}u = \frac{\sin(\ln t)}{t^2}, \qquad 1 \le t \le 2$$
$$u(1) = 1, \quad u'(1) = 0 \tag{12}$$

and

$$v'' + \frac{2}{t}v' - \frac{2}{t^2}v = 0, \quad 1 \le t \le 2$$
$$v(1) = 0, \quad v'(1) = 1 \tag{13}$$

The solution is constructed as $x(t) = u(t) + sv(t)$, where the constant s is determined by enforcing the boundary condition $x(2) = 2$

$$u(2) + sv(2) = 2,$$
$$s = \frac{2 - u(2)}{v(2)}. \tag{14}$$

2. Neumann Boundary Value Problem

Equation (2) with the Neumann boundary conditions $x'(1) = 0$ and $x'(2) = 1$ is solved by considering two auxiliary initial value problems.

$$u'' + \frac{2}{t}u' - \frac{2}{t^2}u = \frac{\sin(\ln t)}{t^2}, \quad 1 \le t \le 2$$
$$u(1) = 0, \quad u'(1) = 0 \tag{15}$$

and

$$v'' + \frac{2}{t}v' - \frac{2}{t^2}v = 0, \quad 1 \le t \le 2$$
$$v(1) = 1, \quad v'(1) = 0 \tag{16}$$

The solution is constructed as $x(t) = u(t) + sv(t)$, where the constant s is determined by enforcing the boundary condition $x'(2) = 1$

$$u'(2) + sv'(2) = 2$$
$$s = \frac{1 - u'(2)}{v'(2)}. \tag{17}$$

3. Robin Boundary Value Problem

Equation (2) with the Robin boundary conditions $x(1) + x'(1) = 1$ and $x(2) + x'(2) = 2$ is solved by considering two auxiliary initial value problems.

$$u'' + \frac{2}{t}u' - \frac{2}{t^2}u = \frac{\sin(\ln t)}{t^2}, \quad 1 \le t \le 2$$
$$u(1) = 1, \quad u'(1) = 0 \tag{18}$$

and

$$v'' + \frac{2}{t}v' - \frac{2}{t^2}v = 0, \quad 1 \le t \le 2$$
$$v(1) = -1, \quad v'(1) = 1 \tag{19}$$

The solution is constructed as $x(t) = u(t) + sv(t)$, where the constant s is determined by enforcing the boundary condition $x(2) + x'(2) = 2$,


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$$u(2) + sv(2) + u'(2) + sv'(2) = 2,$$
$$s = \frac{2 - (u(2) + u'(2))}{v(2) + v'(2)}. \tag{20}$$

Estimated Order of Convergence

To further evaluate the convergence behavior of the numerical methods, the estimated order of convergence (EOC) was computed using successive mesh refinements, following the approach of Gupta & Kaushik (2021). Since the mesh refinements considered in this study were not uniformly doubled, the EOC was evaluated using the generalized expression $p = \frac{ln(E_1/E_2)}{ln(N_2/N_1)}$, where E_1 and E_2 denote the error values corresponding to mesh sizes N_1 and N_2, respectively.

RESULTS AND DISCUSSION

Numerical Result

This section compares the finite difference and linear shooting methods for solving second-order linear homogeneous ordinary differential equations with Dirichlet, Neumann, and Robin boundary conditions. The comparison is based on the error of each method.

Dirichlet Boundary Value Problem

Equation (2) with Dirichlet BVP is

$$\begin{gathered} x''(t) + \frac{2}{t}\, x'(t) - \frac{2}{t^2}\, x(t) = \frac{sin(\ln t)}{t^2}, \\ x(1) = 1, \qquad x(2) = 2. \end{gathered} \tag{21}$$

Applying the Dirichlet boundary conditions in (21) to the general solution (3) gives the exact solution as follows.

$$x(t) = 1.139207013\, t - \frac{0.03920701317}{t^2} - \frac{3}{10}\sin(\ln t) - \frac{1}{10}\cos(\ln t). \tag{22}$$

Table 1. Error Comparison between FDM and LSM for the Dirichlet BVP.

Error	$N = 10$		$N = 20$		$N = 50$	
	FDM	LSM	FDM	LSM	FDM	LSM
MaxAE	$4.55\, x\, 10^{-5}$	$1.34\, x\, 10^{-7}$	$1.14\, x\, 10^{-5}$	$7.67\, x\, 10^{-9}$	$1.83\, x\, 10^{-6}$	$2.80\, x\, 10^{-10}$
MAE	$2.56\, x\, 10^{-5}$	$4.34\, x\, 10^{-8}$	$6.18\, x\, 10^{-6}$	$2.54\, x\, 10^{-9}$	$1.13\, x\, 10^{-6}$	$1.55\, x\, 10^{-10}$

Table 1 presents the error comparison between the FDM and the LSM for the Dirichlet BVP using three different grid sizes, namely $N = 10, N = 20,$ and $N = 50$. The results show that the accuracy of both methods improves as N increases. However, LSM consistently yields smaller MaxAE and MAE values than FDM. For $N = 50$, the MaxAE of LSM is 2.80×10^{-10}, whereas FDM produces an error of 1.83×10^{-6}. These results indicates that LSM provides higher accuracy for the Dirichlet BVP considered in this study.

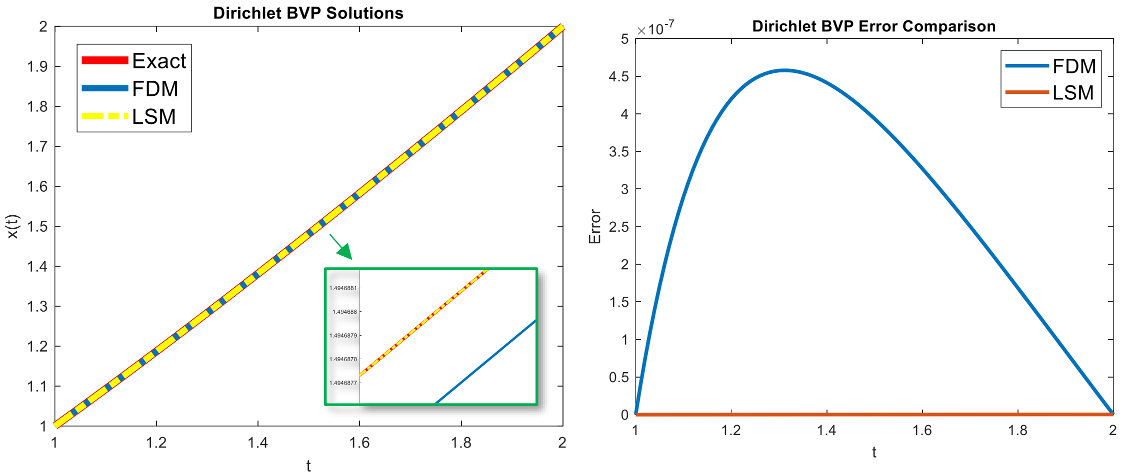

Figure 2. Numerical Results for the Dirichlet BVP with $N = 50$

Figure 2 illustrates the numerical solutions and error distributions of FDM and LSM for $N = 50$. Both methods accurately approximate the exact solution, but LSM exhibits errors that are nearly zero throughout the domain, while FDM produces relatively larger errors. This confirms the superior accuracy of LSM, in agreement with the results reported in Table 1.

Neumann Boundary Value Problem

Equation (2) with Neumann BVP is

$$\begin{gathered} x''(t) + \frac{2}{t}\, x'(t) - \frac{2}{t^2}\, x(t) = \frac{sin(\ln t)}{t^2}, \\ x'(1) = 0, \qquad x'(2) = 1 \end{gathered} \tag{23}$$

Substituting the Neumann boundary conditions in (23) into the general solution (3) yields the exact solution as follows.

$$x(t) \quad = 1.195357454\, t + \frac{0.4476787267}{t^2} - \frac{3}{10}\sin(\ln t) - \frac{1}{10}\cos(\ln t). \tag{24}$$

Table 2. Error Comparison between FDM and LSM for the Neumann BVP.

Error	$N = 10$		$N = 20$		$N = 50$	
	FDM	LSM	FDM	LSM	FDM	LSM
MaxAE	$1.09\,x\,10^{-2}$	$1.37\,x\,10^{-5}$	$2.75\,x\,10^{-3}$	$7.83\,x\,10^{-7}$	$4.41\,x\,10^{-4}$	$1.71\,x\,10^{-8}$
MAE	$6.98\,x\,10^{-3}$	$1.17\,x\,10^{-5}$	$1.74\,x\,10^{-3}$	$6.78\,x\,10^{-7}$	$2.77\,x\,10^{-4}$	$1.50\,x\,10^{-8}$

Table 2 reports the errors of FDM and LSM for the Neumann BVP with $N = 10, N = 20$, and $N = 50$. The results show that increasing the grid size improves the accuracy of both methods. However, LSM consistently achieves lower MaxAE and MAE values than FDM, indicating its better performance for this problem.

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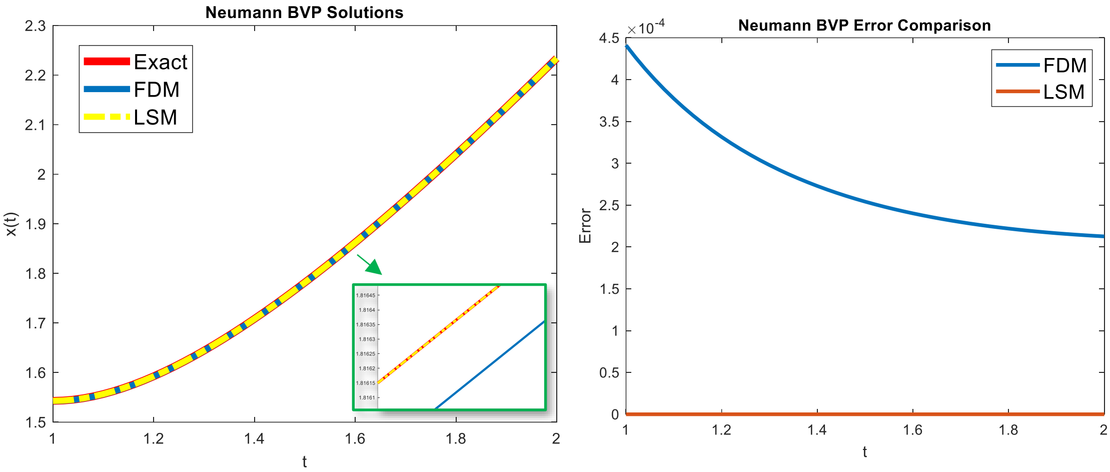

Figure 3. Numerical Results for the Neumann BVP with $N = 50$

Figure 3 depicts the numerical approximations and their corresponding errors for the Neumann BVP with $N = 50$. The solution curves obtained by FDM and LSM are in close agreement with the exact solution across the entire interval. However, a closer inspection of the enlarged region shows that the LSM approximation is almost indistinguishable from the exact solution, while the FDM solution exhibits a larger discrepancy. This difference is also evident from the error plot, where the LSM error remains extremely small throughout the domain, whereas the FDM error is comparatively larger. These findings further support the results presented in Table 2.

Robin Boundary Value Problem

Equation (2) with Robin BVP is

$$x''(t) + \frac{2}{t}\,x'(t) - \frac{2}{t^2}\,x(t) = \frac{\sin(\ln t)}{t^2}$$
$$x(1) + x'(1) = 1, \qquad x(2) + x'(2) = 2. \tag{25}$$

Imposing the Robin boundary conditions in (25) on the general solution (3) produces the exact solution as follows

$$x(t) \;= 0.7840166817\,t + \frac{0.168033363}{t^2} - \frac{3}{10}\sin(\ln t) - \frac{1}{10}\cos(\ln t). \tag{26}$$

Table 3. Error Comparison between FDM and LSM for the Robin BVP.

Error	$N = 10$		$N = 20$		$N = 50$	
	FDM	LSM	FDM	LSM	FDM	LSM
MaxAE	$6.82\ x\ 10^{-3}$	$4.11\ x\ 10^{-6}$	$1.74\ x\ 10^{-3}$	$2.50\ x\ 10^{-7}$	$2.79\ x\ 10^{-4}$	$6.53\ x\ 10^{-9}$
MAE	$3.35\ x\ 10^{-3}$	$1.56\ x\ 10^{-6}$	$8.41\ x\ 10^{-4}$	$8.28\ x\ 10^{-8}$	$1.34\ x\ 10^{-4}$	$1.68\ x\ 10^{-9}$

Leibniz: Jurnal Matematika
Juli 2026
Vol. 6 No. 02 Hal. 01-15
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https://ejurnal.unisap.ac.id/leibniz/index

Table 3 compares the errors produced by FDM and LSM for the Robin boundary value problem with $N = 10, N = 20$, and $N = 50$. The results indicate that finer grids lead to improved accuracy for both methods. However, LSM consistently provides much smaller errors than FDM for all grid sizes. For instance, at $N = 50$, the MaxAE of LSM is 6.53×10^{-9}, whereas FDM yields 2.79×10^{-4}. A similar trend is observed for the MAE values. Therefore, LSM demonstrates superior accuracy in solving the Robin BVP considered in this study.

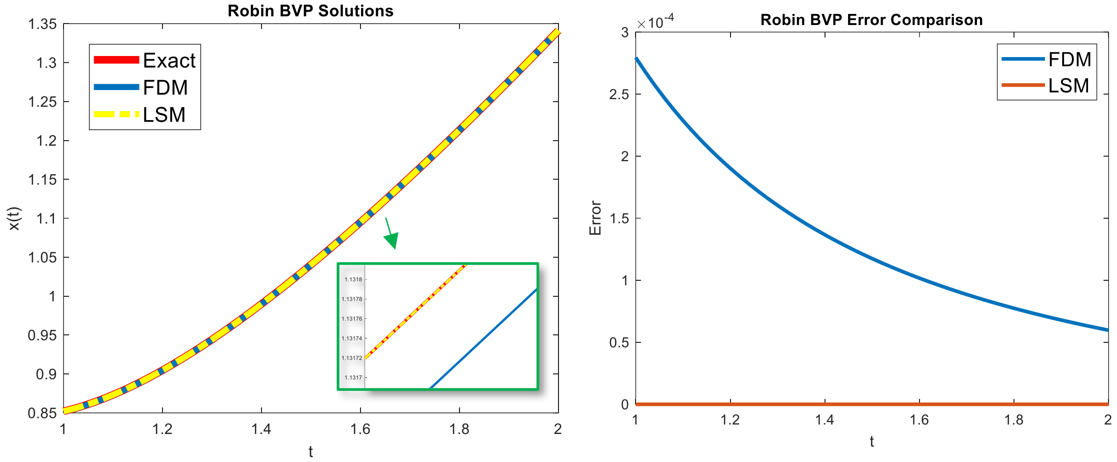

Figure 4. Numerical Results for the Robin BVP with $N = 50$

Figure 4 presents the numerical solutions and error distributions for the Robin BVP with $N = 50$. The numerical solutions obtained by both methods agree well with the exact solution over the entire interval. Nevertheless, the enlarged view shows that the LSM solution is almost identical to the exact solution, while the FDM solution exhibits a more noticeable deviation. The error comparison also reveals that LSM maintains very small errors throughout the domain, whereas FDM produces relatively larger errors. These results are consistent with the error values reported in Table 3.

Estimated Order of Convergence

Table 4. Estimated Order of Convergence for Dirichlet, Neumann, and Robin BVPs.

Method	Order	N Dirichlet BVP		N Neumann BVP		N Robin BVP	
		$(10 \to 20)$	$(20 \to 50)$	$(10 \to 20)$	$(20 \to 50)$	$(10 \to 20)$	$(20 \to 50)$
FDM	p(MaxAE)	1.994	1.998	1.990	1.998	1.973	1.994
	p(MAE)	1.911	1.964	2.004	2.004	1.993	2.003
LSM	p(MaxAE)	4.130	3.611	4.134	4.176	4.041	3.977
	p(MAE)	4.097	3.052	4.115	4.159	4.233	4.256

Table 4 shows that the estimated orders of convergence are consistent across all boundary conditions. The FDM exhibits an estimated convergence order close to two, whereas the LSM

Leibniz: Jurnal Matematika
Juli 2026
Vol. 6 No. 02 Hal. 01-15
e-ISSN: 2775-2356
https://ejurnal.unisap.ac.id/leibniz/index

generally achieves an order close to four. These results indicate that both methods converge as the mesh is refined, with the LSM exhibiting a higher convergence rate than the FDM.

Computational Cost

To complement the accuracy analysis, the computational cost of both numerical methods was evaluated in terms of average execution time.

Table 5. Average Execution Time (in milliseconds) for Dirichlet, Neumann, and Robin BVPs.

Method	Dirichlet BVP			Neumann BVP			Robin BVP		
	$N = 10$	$N = 20$	$N = 50$	$N = 10$	$N = 20$	$N = 50$	$N = 10$	$N = 20$	$N = 50$
FDM	0.067	0.010	0.036	0.311	0.019	0.026	0.411	0.018	0.034
LSM	0.127	0.142	0.282	0.917	0.329	0.591	0.540	0.337	0.449

Table 5 compares the average execution time of the FDM and LSM for different mesh sizes under Dirichlet, Neumann, and Robin BVPs. The FDM requires less execution time than the LSM for all boundary conditions. This is because the FDM directly solves a single system of linear algebraic equations, whereas the LSM solves two equivalent initial value problems using the classical RK4 method before constructing the final solution. Although the LSM incurs a higher computational cost, it consistently produces smaller MaxAE and MAE values than the FDM. These results indicate a trade-off between computational efficiency and numerical accuracy.

Discussion

The numerical results consistently show that the LSM produces smaller MaxAE and MAE values than the FDM for the Dirichlet, Neumann, and Robin BVPs. This higher accuracy is attributed to the numerical formulation of the LSM, which transforms the boundary value problem into equivalent initial value problems and solves them using the classical RK4 method, resulting in a more accurate approximation of the solution. In contrast, the FDM approximates derivatives directly by finite difference formulas, resulting in discretization errors that may accumulate over the computational domain, leading to larger global errors.

The results obtained are in agreement with those reported by Ardiana et al. (2025), who found that the Shooting method generally provides higher accuracy than the Finite Difference Method for solving boundary value problems. However, unlike the present study, Dita Ardiana considered different differential equations for each type of boundary condition. In contrast, this study employs the same Cauchy-Euler differential equation and investigates its behavior under three different boundary conditions, namely Dirichlet, Neumann, and Robin. The present findings are also consistent with those of Habibah et al. (2020), who demonstrated that the Shooting method yields smaller numerical errors than the central Finite Difference Method for the Cauchy-Euler equation with Robin boundary conditions. The present study extends these findings by demonstrating that the superior accuracy of the LSM is not limited to Robin boundary value problems but is also observed for Dirichlet and Neumann boundary conditions.

The estimated orders of convergence further support the error analysis by confirming that the FDM exhibits approximately second-order convergence, whereas the LSM generally achieves a higher convergence order, approaching fourth order in most cases. These findings are consistent with the error analysis, indicating that the LSM provides higher numerical accuracy than the FDM for the boundary value problems considered in this study. Therefore, the LSM can be regarded as a robust and accurate approach for solving second-order Cauchy-Euler BVPs with Dirichlet, Neumann, and Robin boundary conditions.

CONCLUSION

This study compared the performance of the FDM and the LSM in solving a second-order non-homogeneous Cauchy-Euler BVP under Dirichlet, Neumann, and Robin boundary conditions. Both methods produced accurate numerical solutions, with MaxAE and MAE decreasing as the mesh was refined. The estimated orders of convergence confirmed that the FDM exhibits approximately second-order convergence, whereas the LSM generally achieves a higher convergence order. Although the FDM required less computational time, the LSM consistently produced smaller MaxAE and MAE values for all boundary conditions, indicating superior numerical accuracy. Therefore, the LSM may be regarded as a more effective method for solving the non-homogeneous Cauchy-Euler BVP considered in this study. This study is limited to a second-order linear non-homogeneous Cauchy-Euler BVP. Future research may extend this comparison to nonlinear and higher-order BVPs, as well as investigate other numerical methods and additional performance measures.

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